9.3 Fourier Cosine and Sine Series

last time: $f(t)$ period $2L$

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

or any interval $2L$ in length

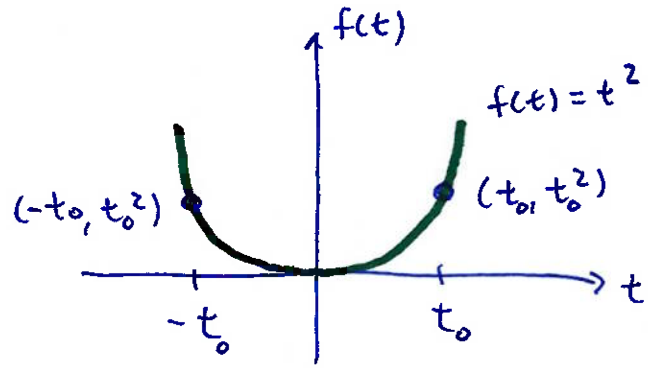
many examples we've seen have purely cosine or sine terms

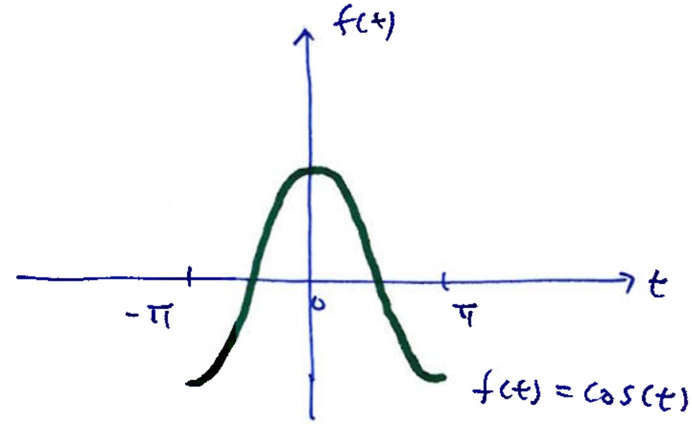
why?

a function $f(t)$ is <u>even</u> if $f(-t) = f(t)$

for example, t^2, t^4, t^6, $\cos(t)$

→ have <u>vertical axis symmetry</u>

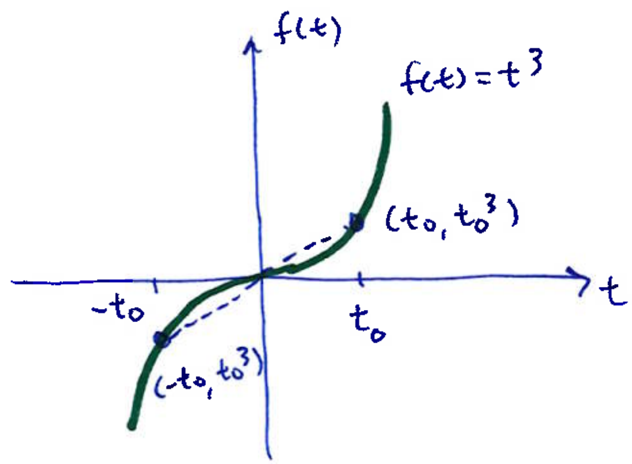


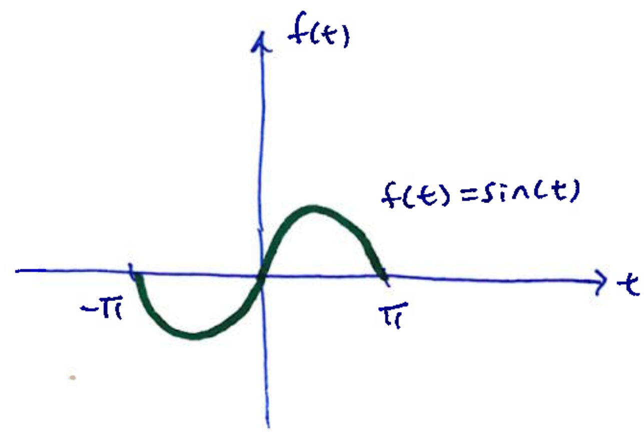


a function $f(t)$ is odd if $f(-t) = -f(t)$

for example, t, t^3, t^5, $\sin(t)$

→ they have origin symmetry





product of two even functions is even

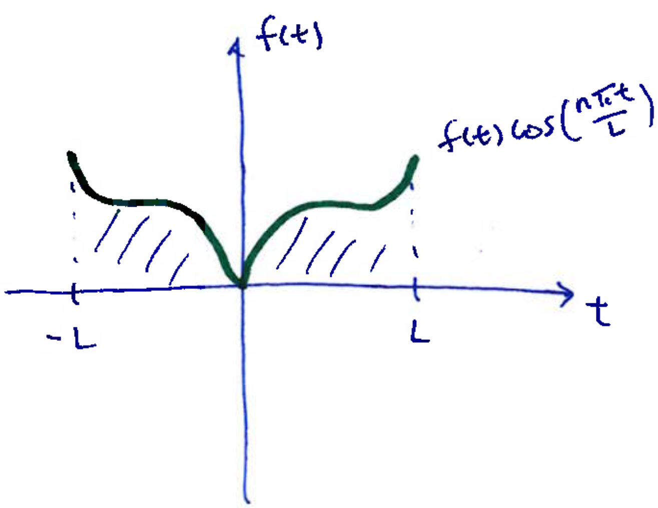
product of two odd functions is even

product of one even and one odd functions is odd

back to $a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(\frac{n\pi t}{L}\right)}_{\text{even}} dt$

if $f(t)$ is even, then $f(t)\cos\left(\frac{n\pi t}{L}\right)$ is even

so, $f(t)\cos\left(\frac{n\pi t}{L}\right)$ has vertical axis symmetry

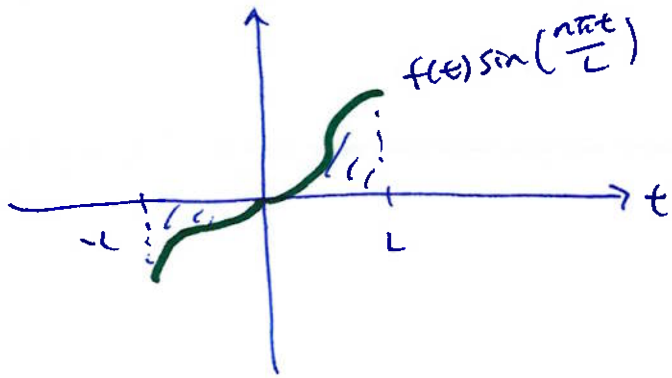


clearly, $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$

$= \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$

$f(t)$ is <u>even</u> , $f(t)\sin\left(\frac{n\pi t}{L}\right)$ is <u>odd</u>

So, $f(t)\sin\left(\frac{n\pi t}{L}\right)$ has origin symmetry



clearly, $b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$

$= 0$ (net area $= 0$)

if $f(t)$ is <u>even</u> w/ period $2L$, then

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$b_n = 0$ for all n

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

Fourier cosine Series

repeating w/ an odd f(t), we see

if $f(t)$ is odd w/ period $2L$, then

$$a_n = 0 \quad \text{for all } n$$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

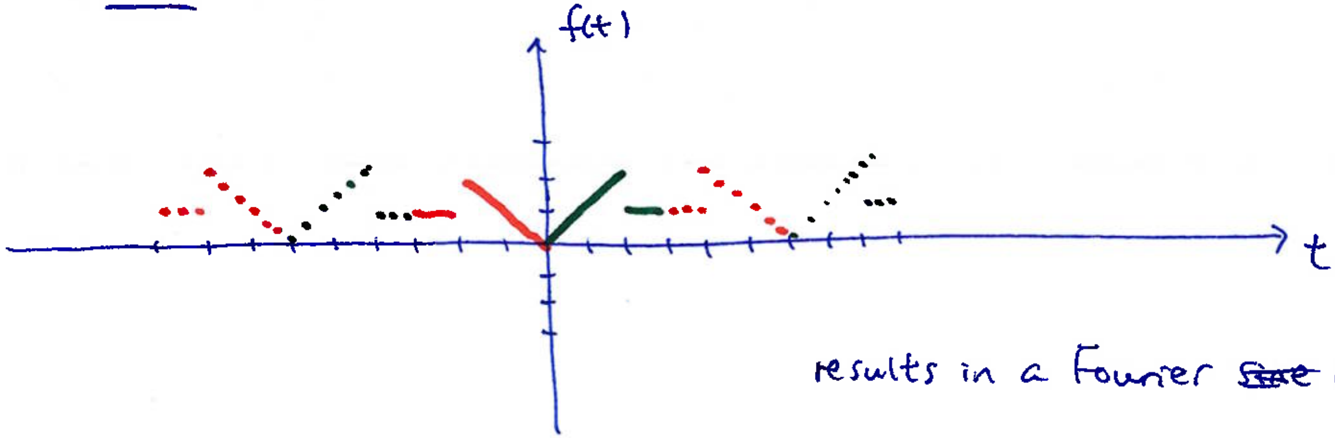
$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

Fourier sine series

we can now describe $f(t)$ in a more compact way by giving $f(t)$ on half a period and specify whether an even extension or an odd extension is added to complete the full period

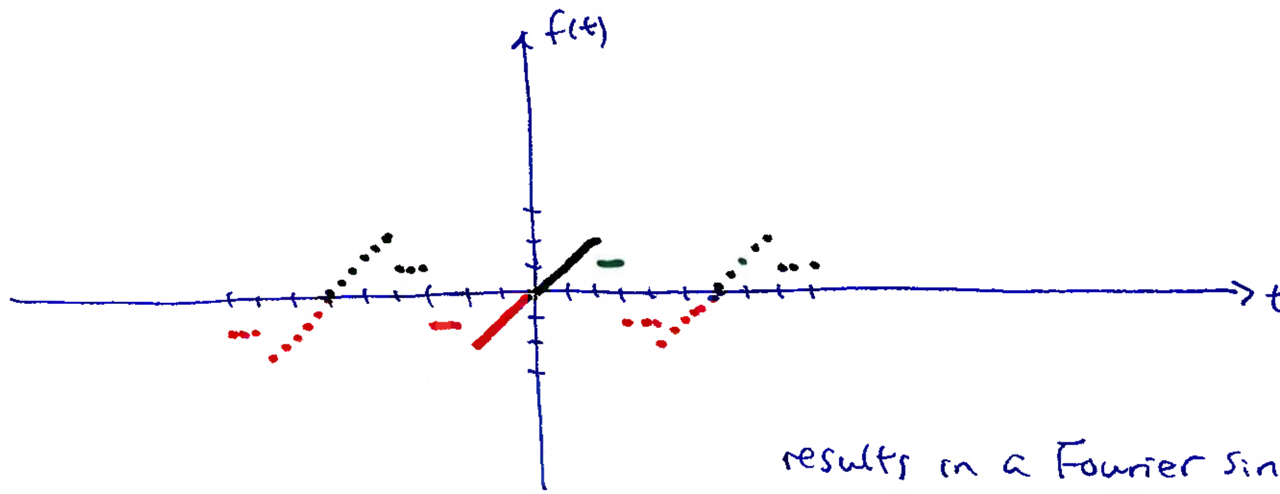
→ tells us whether $f(t)$ is even or odd for example, $f(t) = \begin{cases} t & 0 < t < 2 \\ 1 & 2 < t < 3 \end{cases}$ period is 6

w/ <u>even</u> extensions



results in a Fourier ~~sine~~ cosine series

w/ <u>odd</u> extensions



results in a Fourier sine series

let's write out the Fourier sine series for the 2nd graph

odd: $a_n = 0$

$$b_n = \frac{2}{3}\int_0^3 f(t)\sin\left(\frac{n\pi t}{3}\right)dt$$

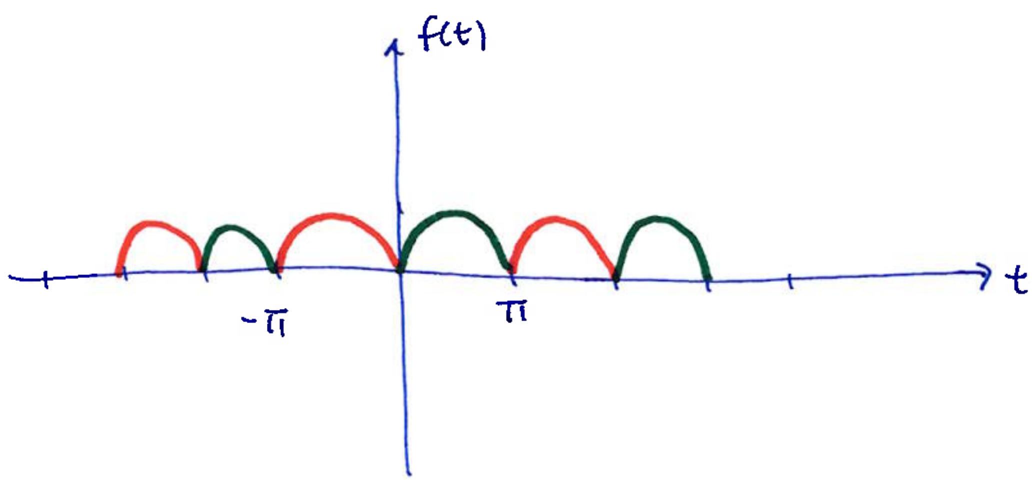
$$= \cdots = \frac{6\sin\left(\frac{2n\pi}{3}\right) - 4n\pi\cos\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} + \frac{2\cos\left(\frac{2n\pi}{3}\right) - 2\cos(n\pi)}{n\pi}$$

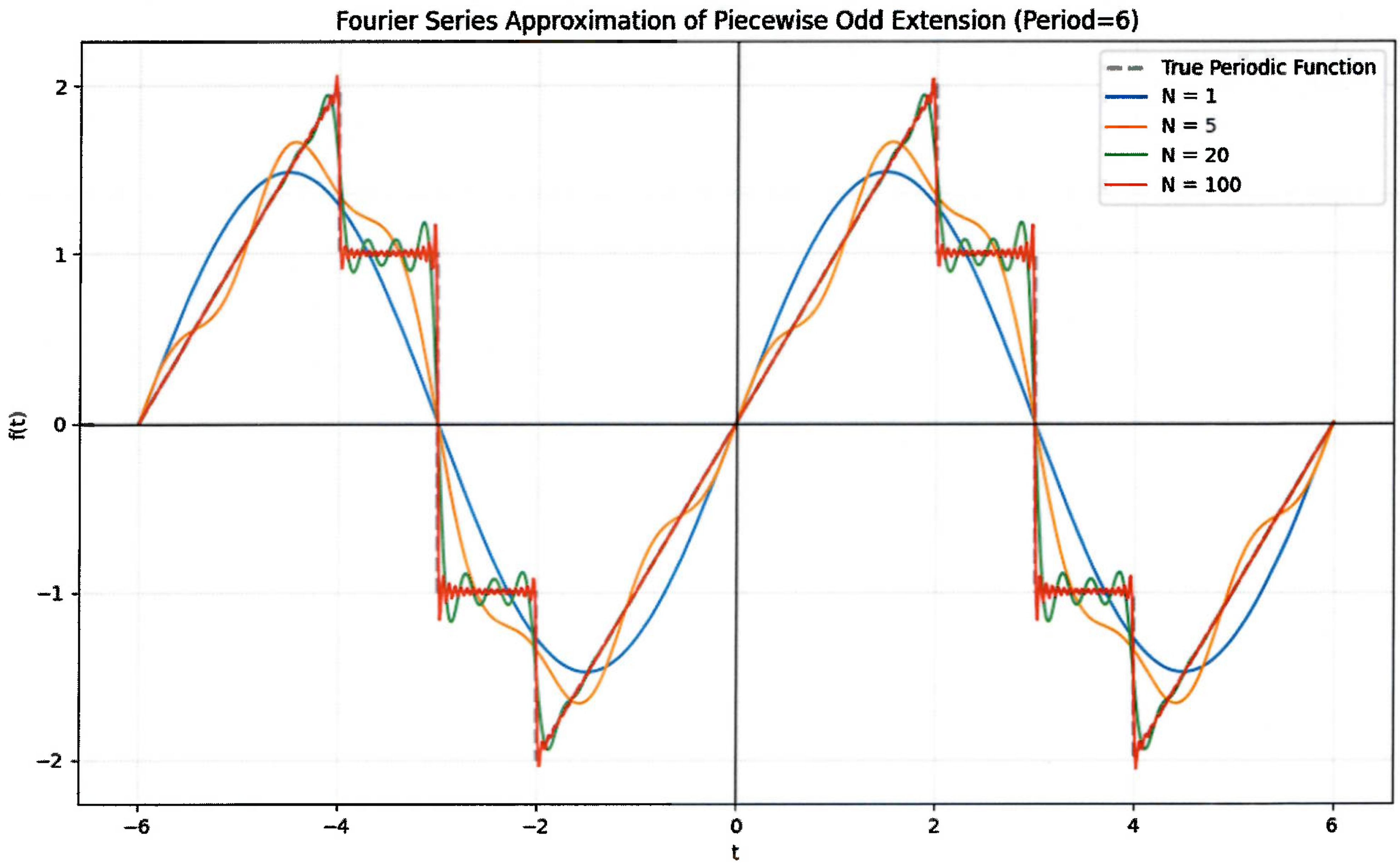
first few terms of the series

$$f(t) \sim \left(\frac{3\sqrt{3}}{\pi^2} + \frac{3}{\pi}\right)\sin\left(\frac{\pi t}{3}\right) + \left(\frac{-3\sqrt{3}}{4\pi^2} - \frac{1}{\pi}\right)\sin\left(\frac{2\pi t}{3}\right) + \cdots$$

another example: $f(t) = \sin(t)$ $\quad 0 < t < \pi$ $\quad$ period 2π w/ even extensions

$\to$ cosine series



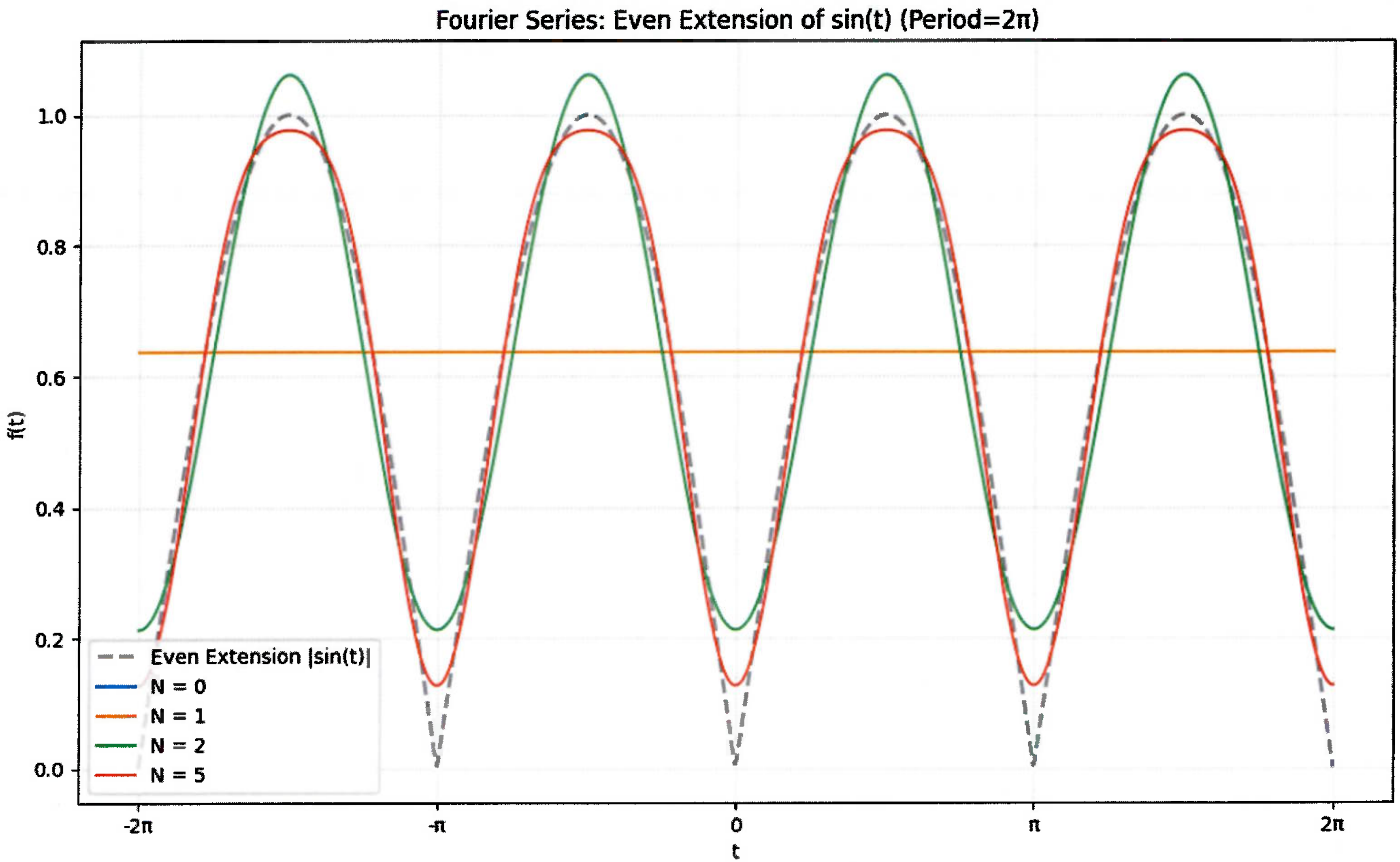

Fourier Series Approximation of Piecewise Odd Extension (Period=6)
True Periodic Function
N = 1
N = 5
N = 20
N = 100
f(t)
t
2
1
0
−1
−2
−6
−4
−2
0
2
4
6

$b_n = 0 \qquad a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{4}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt$$

$$= \cdots = \frac{2\left[1+(-1)^n\right]}{\pi(1-n^2)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{4}{\pi(1-n^2)} & \text{if } n \text{ is even} \end{cases}$$

$$f(t) \sim \frac{2}{\pi} - \frac{4}{3\pi}\cos(2t) - \frac{4}{15\pi}\cos(4t) - \cdots$$


Fourier Series: Even Extension of sin(t) (Period=2π)
1.0
0.8
0.6
0.4
0.2
0.0
f(t)
-2π
-π
0
π
2π
t
Even Extension |sin(t)|
N = 0
N = 1
N = 2
N = 5